6.3 Antidifferentiation by Parts

Product Rule in Integral Form

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

$$\int u\frac{dv}{dx} = \int \frac{d}{dx}uv - \int v\frac{du}{dx}$$

$$= uv - \int v\frac{du}{dx}$$

Integration by Parts Formula

$$\int u\,dv = uv - \underline{\int v\,du}$$

EXAMPLE 1 Using Integration by Parts

Evaluate $\int x \cos x \, dx.$

$(\int u \, dv)$

$u = x \qquad v = \sin x$

$du = dx \qquad dv = \cos x \, dx$

plug into formula

$$= \underset{(u)}{x} \, \underset{(v)}{\sin x} - \int \underset{(v)}{\sin x} \, \underset{(du)}{dx}$$

$$= x \sin x - (-\cos x) + C$$

$$= x \sin x + \cos x + C$$

EXPLORATION 1 **Choosing the Right u and dv**

Not every choice of u and dv leads to success in antidifferentiation by parts. There is always a trade-off when we replace $\int u\,dv$ with $\int v\,du$, and we gain nothing if $\int v\,du$ is no easier to find than the integral we started with. Let us look at the other choices we might have made in Example 1 to find $\int x \cos x\,dx$.

1. Apply the parts formula to $\int x \cos x\,dx$, letting $u = 1$ and $dv = x \cos x\,dx$. Analyze the result to explain why the choice of $u = 1$ is never a good one.
2. Apply the parts formula to $\int x \cos x\,dx$, letting $u = x \cos x$ and $dv = dx$. Analyze the result to explain why this is not a good choice for this integral.
3. Apply the parts formula to $\int x \cos x\,dx$, letting $u = \cos x$ and $dv = x\,dx$. Analyze the result to explain why this is not a good choice for this integral.
4. What makes x a good choice for u and $\cos x\,dx$ a good choice for dv?

1. $u = 1$

$dv = x\cos x\,dx \rightarrow$ can't antidifferentiate

2. $u = x\cos x$ $\quad v = x$

$du = -x\sin x + \cos x\,dx$ $\quad dv = dx$

$$x\cos x \cdot x - \underbrace{\int x\,(-x\sin x + \cos x)\,dx}_{\text{can't do this}}$$

3. $u = \cos x$ $\quad v = \frac{1}{2}x^2$

$du = -\sin x\,dx$ $\quad dv = x\,dx$

$$= \cos\left(\tfrac{1}{2}x^2\right) - \underbrace{\int \tfrac{1}{2}x^2(-\sin x)\,dx}_{\text{can't do this}}$$

EXAMPLE 2 **Repeated Use of Integration by Parts**

Evaluate $\int x^2 e^x \, dx$.

$u = x^2$ $\qquad$ $v = e^x$

$du = 2x\,dx$ $\qquad$ $dv = e^x\,dx$

$$= x^2 e^x - \int e^x (2x)\,dx$$

$u_1 = 2x$ $\qquad$ $v_2 = e^x$

$du_2 = 2dx$ $\qquad$ $dv_2 = e^x\,dx$

$$= x^2 e^x - \left[2xe^x - \int e^x (2dx)\right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

EXAMPLE 3 Solving an Initial Value Problem

Solve the differential equation $\int dy/dx = \int x \ln(x)$ subject to the initial condition $y = -1$ when $x = 1$. Confirm the solution graphically by showing that it conforms to the slope field.

continued

$y =$

$$\int x \ln x \, dx$$

$u = \ln x$

$du = \frac{1}{x} dx$

$v = \frac{1}{2}x^2$

$dv = x\,dx$

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}\cancel{x^2}\left(\frac{1}{\cancel{x}}\right) dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

$$-1 = \frac{1}{2}(1)^2 \ln(1) - \frac{1}{4}(1)^2 + C$$

$$-1 = \frac{1}{2}\ln 1 - \frac{1}{4} + C$$

$$C = -\frac{1}{2}\ln 1 - \frac{3}{4}$$

$\ln 1 \to 0$

$$C = -\frac{3}{4}$$

$$\boxed{y = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 - \frac{3}{4}}$$

EXAMPLE 4 Solving for the Unknown Integral

Evaluate $\int e^x \cos x \, dx$.

$u = e^x$ $\quad v = \sin x$

$du = e^x dx$ $\quad dv = \cos x \, dx$

$$= e^x \sin x - \int \sin x \, e^x \, dx$$

$u_2 = e^x$ $\quad V_2 = -\cos x$

$du_2 = e^x dx$ $\quad dV_2 = \sin x dx$

$$= e^x \sin x - \left[-e^x \cos x - \int -\cos x \, e^x \, dx \right]$$

$$\int e^x \cos x dx = e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x \, dx}_{\text{What we started with}}$$

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C$$

Tabular Integration

EXAMPLE 5 Using Tabular Integration

Evaluate $\int x^2 e^x \, dx$.

SOLUTION

With $f(x) = x^2$ and $g(x) = e^x$, we list:

(u) (dv)

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

Keep going until you get to 0

$$= x^2 e^x - 2xe^x + 2e^x + C$$

EXAMPLE 6 Using Tabular Integration

Evaluate $\int x^3 \sin x \, dx$.

f(x) or u		g(x) or dv
x^3	$+$	$\sin x$
$3x^2$	$-$	$-\cos x$
$6x$	$+$	$-\sin x$
6	$-$	$\cos x$
0		$\sin x$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

EXAMPLE 7 Antidifferentiating ln *x*

Find $\int \ln x \, dx$.

EXAMPLE 8 Antidifferentiating $\sin^{-1} x$

Find the solution to the differential equation $dy/dx = \sin^{-1} x$ if the graph of the solution passes through the point $(0, 0)$.